ankhyā : The Indian Journal of Statistics
1982, Volume 44, Series A, Pt. 1, pp. 72–90.

INTEGRATED CAUCHY FUNCTIONAL EQUATION AND CHARACTERIZATIONS OF THE EXPONENTIAL LAW

KA-SING LAU** and C. RADHAKRISHNA RAO*

University of Pittsburgh

SUMMARY. A general solution of the functional equation $\int_0^\infty f(x+y)d\mu(y) = f(x)$ where f is a nonnegative function and μ is a σ-finite positive Borel measure on $[0, \infty)$ is shown to be $f(x) = p(x) \exp(\lambda x)$ where p is a periodic function with every $y \epsilon \underline{\mu}$, the support of μ as a period. The solution is applied in characterizing Pareto, exponential and geometric distributions by properties of integrated lack of memory, record values, order statistics and conditional expectation.

1. Introduction

The Cauchy functional equation

$$f(x+y) = f(x)f(y) \ \forall \ (x, y) \epsilon \ S \qquad \ldots \quad (1.1)$$

where S is a specified set in R^2 has been extensively studied by a number of authors (see Aczél, 1966). We consider two integrated versions of (1.1),

$$\int_{[0, \infty)} [f(x+y) - f(x)f(y)] d\mu(y) = 0 \ \forall \ x \geqslant x_0 > -\infty, \qquad \ldots \quad (1.2)$$

$$\int_{[0, \infty)} \left[\frac{f(x+y)}{f(y)} - f(x) \right] d\mu(y) = 0 \ \forall \ x \geqslant x_0 > -\infty, \qquad \ldots \quad (1.3)$$

where μ is a positive Borel measure on $[0, \infty)$ and f is nonnegative on $\underline{\mu}$ (the support of μ) in (1.2) and f is positive on $\underline{\mu}$ in (1.3). Both the equations are special cases of a general equation of the type

$$\int_{[0, \infty)} f(x+y) d\mu(y) = f(x) \ \text{a.e. for} \ x \ \epsilon \ [x_0, \infty) \qquad \ldots \quad (1.4)$$

Paper presented at the Conference on Mathematical Statistics and Probability, Indian Statistical Institute, New Delhi, December 1980.

AMS Classification : 39B30, 45A05, 62E10

Key Words : Cauchy functional equation, Choquet-Deny theorem, Exponential distribution, Geometric distribution, Lack of memory, Order statistics, Record values, Pareto distribution.

*The work of C. R. Rao is sponsored by the Air Force Office of Scientific Research under Contract F49620-79-C-0161. Reproduction in whole or in part is permitted for any purpose of the United States Government.

**The work of Ka-Sing Lau is sponsored by NSF MCS-7903638.

where f is nonnegative and μ is as in (1.2) and (1.3). We call (1.4) an Integrated Cauchy Functional Equation (ICFE), or more specifically ICFE(μ) indicating the measure used for integration. We may take x_0 in (1.4) to be zero without loss of generality (as in (2.1) of section 2).

We note that the equation (1.4) occurs in renewal theory with μ as a probability measure and f as a bounded continuous function. In such a case, when μ is not arithmetical, Feller (1971, pp. 364, 382) showed that f is a constant. This result also follows from a general theorem due to Choquet and Deny (1960) which asserts the constancy of a bounded continuous function in a more general situation where f and μ are defined on a locally compact Abelian group.

Recently, a number of papers appeared on the solution of (1.4) under the conditions that f is a locally integrable nonnegative function (or f satisfies a certain growth condition) and that there exists a $\delta > 0$ such that

$$1 < \int_0^\infty e^{2\delta y} \, d\mu(y) < \infty \qquad \ldots \quad (1.5)$$

(see Brandhofe and Davies, 1980; Ramachandran, 1979; Shimizu, 1978 and other papers cited by them). We solve the equation (1.4) without using the stringent condition (1.5), which opens up a wide range of applications. We show that if a nontrivial solution for f exists, then it is of the form

$$f(x) = p(x)e^{\lambda x} \text{ a.e. for } x \geqslant x_0 \qquad \ldots \quad (1.6)$$

where $p(x+z) = p(x) \; \forall \; z \in \underline{\mu}$. The solution is applied to characterize the exponential, Pareto and geometric distributions by lack of memory, properties of record values and order statistics, constancy of conditional expectation etc.

In the special case when $\underline{\mu}$ is a lattice and f is defined only on the support points of μ, the equation (1.4) can be written as

$$\sum_{n=0}^{\infty} f_{m+n} \, g_n = f_m, \quad m = 0, 1, \ldots \qquad \ldots \quad (1.7)$$

where f_m and g_n are nonnegative numbers. An equation of the type (1.7) arose in characterizing a Poisson distribution by what is known as the Rao-Rubin condition based on a damage model introduced by Rao (see Rao, 1965 and Rao and Rubin 1964). Shanbhag (1977) proved that a general solution of (1.7) is of the form

$$f_m = c\beta^m \quad \text{where } \Sigma \, g_i \, \beta^i = 1 \qquad \ldots \quad (1.8)$$

under the conditions that $g_1 \neq 0$ and at least one f_n is not zero. The general result obtained by us provides a complete answer to the problem (1.7) :

(i) If $g_0 > 1$, $f_m = 0 \ \forall \ m$.

(ii) If $g_0 = 1$ and g_i is the first in the series $g_1, g_2, \ldots$ which is not zero, then $f_0, \ldots, f_{i-1}$ are arbitrary and $f_{i+1}, f_{i+2}, \ldots$ all are zero.

(iii) If $g_0 < 1$ and a nontrivial solution exists, then it is of the form

$$f_m = p_m \beta^m \text{ where } \Sigma g_i \beta^i = 1 \qquad \ldots \quad (1.9)$$

where $p_{m+i} = p_m$ if $g_i \neq 0$, thus avoiding the assumption $g_1 \neq 0$.

(iv) If $g_0 < 1$ and no β exists such that $\Sigma \, g_i \beta^i = 1$, then $f_m = 0 \ \forall \ m$ is the only possible solution.

2. Preliminaries

We define the ICFE(μ) as

$$\int_{[0,\infty)} f(x+y) d\mu(y) = f(x) \text{ a.e. } \ x \in [0, \infty) \qquad \ldots \quad (2.1)$$

where μ is a σ-finite positive Borel measure on $[0, \infty)$ and not degenerate at 0, and f is a nonnegative function. We prove a number of lemmas as preliminary to the statement of the main Theorems 3.1 and 3.2 given in section 3.

Lemma 2.1 : *Let f be a locally integrable nonnegative (l.i.n.) solution of the ICFE (μ) and let α be such that*

$$1 \leqslant \int_{[0,\infty)} e^{\alpha y} \, d\mu(y).$$

Then for any $\beta > \alpha$, $x \geqslant 0$

$$\int_x^\infty e^{-\beta y} f(y) dy < \infty. \qquad \ldots \quad (2.2)$$

If we let

$$\tilde{f}(x) = \int_x^\infty e^{-\beta y} f(y) dy \qquad \ldots \quad (2.3)$$

then $\tilde{f}$ is a nonnegative, continuous and decreasing solution of the ICFE($\tilde{\mu}$) where

$$d\tilde{\mu}(y) = e^{\beta y} \, d\mu(y). \qquad \ldots \quad (2.4)$$

Proof : The first statement follows from Theorem 1 of Brandhofe and Davies (1980). The second statement is a consequence of Fubini's theorem.

Lemma 2.2 : *Let f be a l.i.n. solution of* (2.1). *Then the following hold* :

(i) *If* $\mu(0) > 1$, *then* $f \equiv 0$ *a.e.*

(ii) *If* $\mu(0) = 1$, *then* $f = 0$ *a.e.* $\forall\, x \geqslant a$ *where* $a = inf\{x : x \in \underline{\mu}\setminus\{0\}\}$.

(iii) *If* $\mu(0) < 1$, *and there is a solution f which does not vanish a.e., then a strictly positive solution exists. Conversely, if a strictly positive solution f exists, then* $\mu(0) < 1$.

Proof of (i) : Let $\mu(0) > 1$. Then

$$f(x) = \mu(0)f(x) + \int_{(0,\infty)} f(x+y)d\mu(y) \text{ a.e.} \qquad \ldots \quad (2.5)$$

which $\Longrightarrow (\mu(0)-1)f(x) \equiv 0$ a.e. or $f(x) \equiv 0$ a.e.

Proof of (ii). Let $\tilde{f}$ and $\tilde{\mu}$ be as defined in (2.3) and (2.4) respectively. Then $\mu(0) = 1 \Longrightarrow \tilde{\mu}(0) = 1$ and from (2.5) with f and μ replaced by $\tilde{f}$ and $\tilde{\mu}$ we have

$$\int_{(0,\infty)} \tilde{f}(x+y)d\tilde{\mu}(y) = 0. \qquad \ldots \quad (2.6)$$

If $f \not\equiv 0$ a.e. for $x \geqslant a$ (as defined in the statement (ii) of Lemma 2.2), then $\tilde{f}(x) > 0$ for some $x > a$, and because of the decreasing property of $\tilde{f}$, $\tilde{f}(x) > 0$ on a neighbourhood of a. This would imply that (2.6) cannot be zero which is a contradiction.

Proof of (iii). We show that $\tilde{f}$ (which is a continuous decreasing function) cannot vanish. Suppose that $a > 0$ is the smallest value such that $\tilde{f}(x) = 0 \;\forall\, x \geqslant a$. Choose $0 < \delta < a$ such that $\tilde{\mu}(0, \delta) < 1 - \tilde{\mu}(0)$.

Then
$$\begin{aligned}\tilde{f}(a-\delta) &= \tilde{f}(a-\delta)\tilde{\mu}(0) + \int_{(0,\delta]} \tilde{f}(a-\delta+y)d\tilde{\mu}(y)\\ &< \tilde{f}(a-\delta)\tilde{\mu}(0) + \tilde{f}(a-\delta)(1-(\tilde{\mu}(0)) = \tilde{f}(a-\delta)\end{aligned}$$

which is a contradiction. The result is established by noting that $\tilde{f}(0) \neq 0$ and $h(x) = e^{\beta x}\tilde{f}(x)$ is a positive solution of ICFE(μ). The second part of statement (iii) follows immediately from (i) and (ii).

For any subset A in $[0, \infty)$, let

$$A_n = \{x_1 + \ldots + x_n : x_i \in A,\; i = 1, \ldots, n\}.$$

If μ^n denotes the n-fold convolution of μ, then $\underline{\mu^n} = \underline{\mu_n}$, using the notation $\underline{\mu}$ to denote the support of μ. Let

$$G = \{x - y : x, y \in \bigcup_{n=1}^{\infty} \underline{\mu_n}\}.$$

Then G is a subgroup of R and is either dense in R or discrete in R.

Lemma 2.3 : *Let $0 < \delta < 1$ and, $\underline{\mu}$, E and G be as defined above. Then there exist a finite set $A = \{p_1, \ldots, p_m\} \subseteq E$ with each $p_i > 1$ and an x_0 such that for any $x \in G$ with $x > x_0$, we can find a $y \in \bigcup A_n \subseteq \bigcup \underline{\mu_n}$ satisfying the condition $|y - x| < \delta$.*

Proof: Consider the case where $\underline{\mu}$ generates a discrete group G which may be taken as N (the set of integers) without loss of generality. Hence there exists a finite subset $A = \{p_1, \ldots, p_m\}$ in $\underline{\mu}$ such that the greatest common divisor of $p_1, \ldots, p_m$ is unity. We prove the assertion for $m = 2$ in which case the general result follows by induction. It is easily seen that for $m > p_1 p_2$ there exist $\alpha, \beta \in N^+$ (the set of positive integers) such that $m = \alpha p_1 + \beta p_2$. Then the result of the lemma holds since $m \in G$ and $(\alpha p_1 + \beta p_2) \in \bigcup A_n$.

Now consider the case where $\underline{\mu}$ generates a dense subgroup in R. There are two possibilities :

Case (*a*). There exists a countable infinite subset $B = \{p_1, p_2, \ldots\} \subseteq \underline{\mu}$ which generates a dense subgroup of R and that for each n, $\{p_1, \ldots, p_n\}$ generates a discrete subgroup $\{md_n : m \in N\}$ with $d_n \geqslant 0$. Then $\lim d_n = 0$ as $n \to \infty$ and hence, for any $\delta > 0$ we can find an n_0 such that $d_{n_0} < \delta$. Let $A = \{p_1, \ldots, p_{n_0}\}$. It follows from the previous part that there exists an x_0 such that for $x > x_0$, there exists a $y \in \bigcup A_n \subseteq \bigcup \underline{\mu_n}$ and $|y - x| < \delta$.

Case (*b*). There exist p_1, p_2 such that they generate a dense subgroup in R. Without loss of generality, we may assume that $p_1 = p$ is an irrational number and $p_2 = 1$ and let $A = \{1, p\}$. It is well known that the set $D = \{np - [np] : n \in N^+\}$ is dense in $[0, 1]$. For any $\delta > 0$, there exist $n_1 p, \ldots, n_k p$ such that for any $\xi \in [0, 1]$,

$$|n_i p - [n_i p] - \xi| < \delta \text{ for some } i.$$

Let $n_0 = \max\{[n_i p] : i = 1, \ldots, k\}$. Then for any $x > n_0$, there exists an i such that

$$|n_i p - [n_i p] - (x - [x])| = |\{n_i p + [x] - [n_i p]\} - x| < \delta$$

with $n_t > 0$ and $[x]-[n_t p] \geqslant 0$. This completes the proof since

$$n_t p+([x]-[n_t p]) \in \bigcup A_n \subseteq \bigcup \underline{\mu}_n.$$

3. The main theorems

The following theorems provide a solution of (1.4), the ICFE(μ).

Theorem 3.1 : *Let f be a positive locally integrable solution of the ICFE(μ). Then*

$$f(x) = p(x)e^{\lambda x} \text{ a.e.} \quad \ldots \quad (3.1)$$

where $\lambda \in (-\infty, \infty)$ and is uniquely determined by

$$1 = \int_0^\infty e^{\lambda y}\, d\mu(y) \quad \ldots \quad (3.2)$$

and p is a positive periodic function with every $z \in \underline{\mu}$ as a period.

Theorem 3.2 : *Let f be a nonnegative locally integrable solution of the ICFE(μ). If $\mu(0) < 1$, then $f(x) = p(x)exp(\lambda x)$ a.e. where λ satisfies* (3.2) *and p is nonnegative and periodic with every $z \in \underline{\mu}$ as a period.*

Note that if there exists no λ such that (3.2) holds, then $f = 0$, a.e. is the only solution.

Remark 1 : It suffices to prove Theorem 3.1 for f positive, convex and decreasing.

Let β be as in Lemma 2.1.

Then

$$h(x) = \int_x^\infty \left(\int_y^\infty e^{-2\beta t} f(t)dt \right) dy$$

is convex, decreasing and satisfies ICFE($\tilde{\mu}$) where $d\tilde{\mu}(y) = e^{2\beta y}\, d\mu(y)$. If μ is proved to be of the form $\mu(x) = p(x)e^{\lambda x}$, then f will also be of the same form by taking derivatives.

Remark 2 : It suffices to prove Theorem 3.1 for $\nu = \sum_1^\infty 2^{-n}\mu^n$ where μ^n is the n-fold convolution of μ.

This follows from the fact that if f is a solution of the ICFE (μ), then it is also a solution of the ICFE (ν), by using Fubini's theorem. We use the property that $\underline{\nu}' \supseteq$ the semigroup $\bigcup \underline{\mu}_n$.

Remark 3 : Let $E = \{x \in \underline{\mu} : \mu(x-\delta, x] > 0 \ \forall\ \delta > 0\}$. It is easy to show that $\underline{\mu} \setminus E$ is a μ-zero set and E is dense in $\underline{\mu}$. It suffices to prove the periodicity of p for $y \in E$.

Indeed, by Remark 1 we can assume that f is continuous. Let $f(x) = p(x)e^{\lambda x}$ where $p(x+y) = p(x)$, $\forall\ y \in E$. Since E is dense in $\underline{\mu}$, the continuity of p implies that p is periodic for all $y \in \bar{E}$ which is the same as $\underline{\mu}$.

Since the proof is quite long, we will divide it into several lemmas and a main proof. We assume that f is as in Remark 1, the measure μ is of the form ν mentioned in Remark 2, and $1 \in E$ where E is as in Remark 3. The same proof works for any α. We prove that $f(x) = p(x)e^{\lambda x}$ where p is a periodic function with period 1. We denote

$$c = \sup_{[x \in 0, \infty)} \frac{f(x+1)}{f(x)} \leqslant 1.$$

Lemma 3.1 : *Let δ be such that $0 < \delta < c$. Then*

$$\delta \leqslant c - \frac{f(x+1)}{f(x)} \implies \frac{\delta}{2} \leqslant c - \frac{f(x+y+1)}{f(x+y)} \ \forall\ y \in \left[-\frac{\delta}{2}, 0\right]. \quad \dots \quad (3.3)$$

Proof : By the convexity of f we have for $-1 < y \leqslant 0$, and any x

$$f(x+y+1) \leqslant f(x) + [f(x+1) - f(x)](y+1)$$

and

$$f(x+y) \geqslant f(x) + [f(x+1) - f(x)]y.$$

Hence

$$\frac{f(x+y+1)}{f(x+y)} \leqslant \frac{f(x) + [f(x+1) - f(x)](y+1)}{f(x) + [f(x+1) - f(x)]y}$$

$$\leqslant \frac{1 + (c-\delta-1)(y+1)}{1 + (c-\delta-1)y} \quad \dots \quad (3.4)$$

if x is such that $f(x+1) \leqslant (c-\delta)f(x)$. The inequality

$$\frac{1+(c-\delta-1)(y+1)}{1+(c-\delta-1)y} > c - \frac{\delta}{2} \implies -y > \frac{\delta/2}{(1-c+\delta)(1-c+\delta/2)} > \frac{\delta}{2}.$$

Hence for $-\delta/2 \leqslant y \leqslant 0$, from (3.4)

$$\frac{f(x+y+1)}{f(x+y)} \leqslant \frac{1+(c-\delta-1)(y+1)}{1+(c-\delta-1)y} \leqslant c-\frac{\delta}{2}.$$

Lemma 3.2 : *For any $b > a$, $x \geqslant 0$,*

$$0 \leqslant \int_{(a,\,b]} \left(c-\frac{f(x+y+1)}{f(x+y)}\right) \frac{f(x+y)}{f(x)}\, d\mu(y) \leqslant c-\frac{f(x+1)}{f(x)}. \quad \dots \quad (3.5)$$

Proof : Since f satisfies the ICFE(μ)

$$\frac{f(x+1)}{f(x)} = \int_0^\infty \frac{f(x+y+1)}{f(x)}\, d\mu(y)$$

$$\leqslant \int_{(a,\,b]}^{\infty} \frac{f(x+y+1)}{f(x+y)} \frac{f(x+y)}{f(x)}\, d\mu(y)+c$$

$$-c \int_{(a,\,b]} \frac{f(x+y)}{f(x)}\, d\mu(y)$$

$$= c+\int_{(a,\,b]} \left(\frac{f(x+y+1)}{f(x+y)}-c \right) \frac{f(x+y)}{f(x)}\, d\mu(y)$$

which yields the desired result.

Lemma 3.3 : *Let $0 < \varepsilon_0 < c$ and k be a given positive integer. Then there exists an ε_1 such that*

$$0 < c-\frac{f(x+1)}{f(x)} < \varepsilon_1 \Longrightarrow 0 < c-\frac{f(x+n+1)}{f(x+n)} < \varepsilon_0 \;\forall\; 0 \leqslant n \leqslant k. \quad \dots \quad (3.6)$$

Proof : Let $\eta_k = \varepsilon_0/2$ and choose $\eta_{k-1}, \dots, \eta_0$ such that $0 < \eta_n < \varepsilon_0/2$ and

$$0 < \frac{2\eta_n}{(c-2\eta_n)\mu(1-\eta_{n+1},\, 1]} < \eta_{n+1},\ \ n = 0, \dots, k-1.$$

[Note that $1 \epsilon E$ by assumption so that $\mu(1-\eta_{i+1}, 1] > 0$]. Now let $\varepsilon_1 = 2\eta_0$. Then for any x with $0 < c-[f(x+1)/f(x)] < 2\eta_0$. Lemma 3.2 implies

$$\int_{1-\eta_1}^{1} \left(c-\frac{f(x+y+1)}{f(x+y)}\right) \frac{f(x+y)}{f(x)}\, d\mu(y) \leqslant c-\frac{f(x+1)}{f(x)} \leqslant 2\eta_0.$$

Applying the mean value theorem to the above integral

$$c-\frac{f(x+y'+1)}{f(x+y')} \leqslant \frac{f(x)}{f(x+y')}\cdot\frac{2\eta_0}{\mu(1-\eta_1, 1]} \text{ for some } 1-\eta_1 < y' \leqslant 1$$

$$\leqslant \frac{f(x)}{f(x+1)}\cdot\frac{2\eta_0}{\mu(1-\eta_1, 1]}$$

$$\leqslant \frac{2\eta_0}{(c-2\eta_0)\mu(1-\eta_1, 1]} \leqslant \eta_1.$$

By using a contradiction argument of Lemma 3.1 (replacing x in the lemma by $x+1$)

$$0 < c-\frac{f(x+2)}{f(x+1)} \leqslant 2\eta_1 < \varepsilon_0.$$

By repeating the same argument,

$$0 < c-\frac{f(x+n+1)}{f(x+n)} \leqslant 2\eta_n < \varepsilon_0 \;\forall\; 0 \leqslant n \leqslant k.$$

Lemma 3.4 : *Let* $p \in E \subseteqq \mu$. *Then*

(i) $c^p = b$ *where* $b = \sup_x \frac{f(x+p)}{f(x)}$. ... (3.7)

(ii) *For any integer* $k > 0$ *and any* $\varepsilon_0 > 0$, *there exists an* ε_1 *such that*

$$\frac{f(x+p)}{f(x)} > (c-\varepsilon_1)^p \Longrightarrow \frac{f(x+\overline{n+1}p)}{f(x+np)} > (c-\varepsilon_0)^p \;\forall\; 1 \leqslant n \leqslant k. \quad \text{... (3.8)}$$

(iii) *If* $p > 1$, *then for any* $\varepsilon_0 > 0$, *there exists an* ε_2 *such that*

$$\frac{f(x+1)}{f(x)} > c-\varepsilon_2 \Longrightarrow \frac{f(x+p)}{f(x)} > (c-\varepsilon_0)^p. \quad \text{... (3.9)}$$

Proof of (i) : For any $\varepsilon_0 > 0$, there exist integers $m, n > 0$ such that $n-\varepsilon_0 \leqslant mp \leqslant n$. From Lemma 3.3, there exists an η for given n such that

$$\frac{f(x+1)}{f(x)} > (c-\eta) \Longrightarrow \frac{f(x+n)}{f(x)} > (c-\varepsilon_0)^n.$$

Hence

$$b^m \geqslant \frac{f(x+mp)}{f(x)} \geqslant \frac{f(x+n)}{f(x)} > (c-\varepsilon_0)^n \geqslant (c-\varepsilon_0)^{mp+\varepsilon_0}$$

which implies that $b \geqslant (c-\varepsilon_0)^{p+\varepsilon_0/m}$. Since ε_0 is arbitrary, we have $b \geqslant c^p$.

Note that Lemma 3.3 will apply for p in the place of 1; we then have

$$\frac{f(x+p)}{f(x)} > b-\eta \Longrightarrow \frac{f(x+mp)}{f(x)} > (b-\varepsilon_0)^m.$$

Now choose $m, n > 0$ such that $mp-\varepsilon_0 \leqslant n \leqslant mp$.

Then $$c^n \geqslant \frac{f(x+n)}{f(x)} \geqslant \frac{f(x+mp)}{f(x)} > (b-\varepsilon_0)^m$$

which $\Longrightarrow$ $c^p \geqslant b$. Hence $b = c^p$ which establishes (i).

Proof of (ii) : Again considering p for 1 in Lemma 3.3, and observing that $b = c^p$, we obtain the result (ii).

Proof of (iii) : Let $\varepsilon_0 > 0$ and m, n be such that $n-\varepsilon_0 \leqslant mp \leqslant n$. Let

$$0 < \eta' < c-[c^{p(m-1)}(c-\varepsilon_0)^p]^{1/(mp+\varepsilon_0)}.$$

(Note that the condition $p > 1$ ensures that the term on the right hand side is positive for small ε_0). Applying Lemma 3.3, there exists ε_2 such that

$$\frac{f(x+1)}{f(x)} > c-\varepsilon_2 \Longrightarrow \frac{f(x+n)}{f(x)} > (c-\eta')^n.$$

Then

$$\frac{f(x+mp)}{f(x)} \geqslant \frac{f(x+n)}{f(x)} > (c-\eta')^n \geqslant (c-\eta')^{mp+\varepsilon_0}.$$

We claim that

$$\frac{f(x+p)}{f(x)} > (c-\varepsilon_0)^p,$$

for otherwise

$$(c-\eta')^{mp+\varepsilon_0} \leqslant \frac{f(x+mp)}{f(x+\overline{m-1}p)} \cdots \frac{f(x+p)}{f(x)}$$

$$\leqslant c^{(m-1)p}(c-\varepsilon_0)^p$$

which contradicts the choice of η'. Thus result (iii) is proved.

Lemma 3.5 : *Let* $0 < \varepsilon_0 < c$ *and* $p_1, \ldots, p_n$ *and* m_0 *be chosen as in Lemma* 2.3 (*with all* $p_i > 1$). *Then for any* $a > 0$, *there exists an* ε_1 *such that*

$$\frac{f(x+1)}{f(x)} > c-\varepsilon_1 \Longrightarrow \frac{f(x+m_0+y)}{f(x+m_0)} > (c-\varepsilon_0)^{y+\varepsilon_0} \qquad \ldots \quad (3.10)$$

for all $y \in [0, a] \cap G$.

Proof : For simplicity, we will prove the case $n = 1$, and the extension to general n can be carried out in a similar way.

Let us denote p_1 by p. There are finitely many m's and n's such that $(n+mp) \in [m_0, m_0+a]$, say k. Choose $\eta_0 \in (0, \varepsilon_0)$ such that

$$\frac{(c-\eta_0)^{mp}}{c^{m_0-n}} > (c-\epsilon_0)^{(n+mp)-m_0} \;\forall\; n+mp \in [m_0, m_0+a]. \qquad \ldots \quad (3.11)$$

By Lemma 3.4, (ii), we can find $\eta_1 \in (0, \eta_0)$ such that

$$\frac{f(x+p)}{f(x)} > (c-\eta_1)^p \Longrightarrow \frac{f(x+\overline{m+1}p)}{f(x+mp)} > (c-\eta_0)^p, \; 1 \leqslant m \leqslant k.$$

By Lemma 3.4, (iii), we can find $\eta_2 \in (0, \eta_1)$ such that

$$\frac{f(x+1)}{f(x)} > (c-\eta_2) \Longrightarrow \frac{f(x+p)}{f(x)} > (c-\eta_1)^p.$$

By Lemma 3.3 we can find $\varepsilon_1 \in (0, \eta_2)$ such that

$$\frac{f(x+1)}{f(x)} > (c-\varepsilon_1) \Longrightarrow \frac{f(x+n+1)}{f(x+n)} > c-\eta_2 \;\forall\; 1 \leqslant n \leqslant k.$$

It follows that for $n+mp \in [m_0, m_0+a]$,

$$\frac{f(x+1)}{f(x)} > c-\varepsilon_1$$

$$\Longrightarrow \frac{f(x+n+mp)}{f(x+m_0)} = \frac{f(x+n+mp)}{f(x+n)} \cdot \frac{f(x+n)}{f(x+m_0)}$$

$$> (c-\eta_0)^{mp+n-m_0} \quad \text{if } n \geqslant m_0, \qquad \ldots \quad (3.12)$$

$$> (c-\eta_0)^{mp}/c^{m_0-n} \quad \text{if } n < m_0. \qquad \ldots \quad (3.13)$$

In the case of (3.12), and in the case of (3.13) by the construction (3.11), we have

$$\frac{f(x+n+mp)}{f(x+m_0)} > (c-\varepsilon_0)^{n+mp-m_0}.$$

Now for any $y \in [0, a] \cap G$, there exists $n+mp$ such that

$$n+mp-\varepsilon_0 \leqslant m_0+y \leqslant n+mp.$$

Hence

$$\frac{f(x+m_0+y)}{f(x+m_0)} \geqslant \frac{f(x+n+mp)}{f(x+m_0)} > (c-\varepsilon_0)^{n+mp-m_0} \geqslant (c-\varepsilon_0)^{y+\varepsilon_0}$$

which completes the proof.

Main proof of Theorem 3.1 : From Lemma 3.4(i), $[f(x+y)/f(x)] \leqslant c^y \;\forall\; y \in E$. Suppose that $[f(x+1)/f(x)] < c$ for some x. By the continuity of f, it follows that $[f(x+y)/f(x)] < c^y \;\forall\; y$ in a neighbourhood of 1. Hence

$$1 = \int_0^\infty \frac{f(x+y)}{f(x)}\, d\mu(y)$$

$$< \int_0^\infty c^y \, d\mu(y).$$

(Strict inequality holds since we assume that $\mu(1-\delta, 1] > 0$ for $\delta > 0$). Then there exist a and ε_0 such that

$$1 < \int_{[0,\, a]} (c-\varepsilon_0)^{y+\varepsilon_0}\, d\mu(y).$$

For the above a and ε_0, we take m_0 and ε_1 as in Lemma 3.5 and choose x_0 such that $[f(x_0+1)/f(x_0)] > c-\varepsilon_1$. Hence

$$1 < \int_{[0,\, a]} (c-\varepsilon_0)^{y+\varepsilon_0}\, d\mu(y)$$

$$< \int_{[0,\, a] \cap G} \frac{f(x_0+m_0+y)}{f(x_0+m_0)}\, d\mu(y), \text{ by Lemma 3.5}$$

$$\leqslant \int_0^\infty \frac{f(x_0+m_0+y)}{f(x_0+m_0)}\, d\mu(y) = 1,$$

This is a contradiction and hence $[f(x+1)/f(x)] = c$ for all x. This implies that there exists a λ such that $f(x) = p(x)e^{\lambda x}$ where p is a periodic function with period 1. The above argument applies to any $y \in E$ instead of 1. We then have $f(x) = q(x)\, e^{\alpha x}$ where q is a periodic function with period y. Hence $\lambda = \alpha$, and $p = q =$ constant on the set $a+D_1$ where $a \in [0, \infty)$ and

$$D_1 = \{x > 0 : x = m \cdot 1 + ny,\ m,n \in N\}.$$

We conclude that $f(x) = p(x)\, e^{\lambda x}$ for some λ where p is constant on the set $a+D$ with $a \in [0, \infty)$ and

$$D = \left\{x > 0 : x = \sum_1^n m_i y_i,\ m_i \in N^+,\ y_i \in E,\ i = 1, \ldots, n,\ n \in N^+\right\}$$

where N^+ is the set of positive integers, which implies that $f(x) = p(x)\, e^{\lambda x}$ where p is a periodic function with every $y \in E$ as a period.

To prove

$$\int_0^\infty e^{\lambda y}\, d\mu(y) = 1$$

we observe that

$$p(x)\, e^{\lambda x} = f(x) = \int_0^\infty p(x+y)\, e^{\lambda(x+y)}\, d\mu(y)$$

$$= \int_{\underline{\mu}} p(x+y)\, e^{\lambda(x+y)}\, d\mu(y) = p(x)\, e^{\lambda x} \int_{\underline{\mu}} e^{\lambda y}\, d\mu(y)$$

which yields the desired result.

Theorem 3.2 follows by using Lemma 2.2 and Theorem 3.1.

4. Characterizations of the exponential law

4.1. *Lack of memory* : A nonnegative random variable X is said to have the lack of memory property if

$$P(X > x+y) = P(X > x)P(X > y)\ \forall\ (x, y) \in S \subsetneq R^2. \quad \ldots (4.1.1)$$

Denoting $F(x) = P(X \leqslant x)$ and $G(x) = 1-F(x)$, the condition (4.1.1) is equivalent to

$$G(x+y) = G(x)G(y), \quad \ldots (4.1.2)$$

or

$$\frac{G(x+y)}{G(y)} = G(x), \quad \text{when } G(y) \neq 0. \quad \ldots (4.1.3)$$

Under some conditions on the set S, it is shown that (4.1.1) $\Longrightarrow X \sim E(\lambda)$, i.e., follows the exponential distribution (see Galambos and Kotz, 1978 for complete bibliography and detailed proofs). Recently, Huang (1978), Ramachandran (1979) and Shimizu (1978) considered an integrated version of (4.1.2)

$$\int_{[0,\infty)} G(x+y)d\mu(y) = G(x) \int_{[0,\infty)} G(y)d\mu(y) = cG(x) \qquad \ldots \quad (4.1.4)$$

where μ is a p.d.f. and showed that if μ is not degenerate and $\mu(0) < c$, then G is of the form

$$G(x) = p(x)e^{-\lambda x} \qquad \ldots \quad (4.1.5)$$

where $p(x+z) = p(x)$ for every $z \in \underline{\mu}$, the support of μ. In particular if $\underline{\mu}$ is not lattice, then $G(x) = e^{-\lambda x}$ i.e., $X \sim E(\lambda)$.

We consider an integrated version of (4.1.3)

$$\int_{[0,\infty)} \frac{G(x+y)}{G(y)} d\mu(y) = G(x) \;\forall\; x \geqslant x_0 \qquad \ldots \quad (4.1.6)$$

where μ is a p.d.f. and $G(y)$ is positive on $\underline{\mu}$. If we write $d\mu_*(y) = [G(y)]^{-1}d\mu(y)$, then the equation (4.1.6) can be written

$$\int_{[0,\infty)} G(x+y)d\mu_*(y) = G(x). \qquad \ldots \quad (4.1.7)$$

Since μ_* need not be a p.d.f., the results of the earlier authors do not apply. However, Theorem 3.2 shows that

$$G(x) = p(x)e^{-\lambda x} \qquad \ldots \quad (4.1.8)$$

provided that $\mu(0) < G(0)$, where $p(x+z) = p(x)$ for every $z \in \underline{\mu_*} = \underline{\mu}$. If the support of μ is not a lattice, then p is a constant. From Theorem 3.1, λ satisfies the equation

$$1 = \int_{[0,\infty)} e^{-\lambda y}d\mu_*(y) = \int_{[0,\infty} e^{-\lambda y}(pe^{-\lambda y})^{-1}d\mu(y)$$

which is true for any λ with $p = 1$. Of course $\lambda > 0$, since $1-G$ is a p.d.f. Thus, the equation (4.1.6) provides a characterization of $E(\lambda)$.

Let us consider a discrete version of the equation (4.1.6) with the random variable X taking values $0, 1, \ldots$ with probabilities $p_0, p_1, \ldots$. Defining $P_m = p_{m+1}+p_{m+2}+\ldots$, and assuming that $P_m \neq 0$, $m = 0, 1, \ldots$, the equation becomes

$$\sum_{n=0}^{\infty} \frac{P_{m+n}q_n}{P_n} = P_m, \quad m = 0, 1, \ldots. \qquad \ldots \quad (4.1.9)$$

Applying the result (1.9), or the general Theorem 3.1,

$$P_m = c_m\beta^m \qquad \ldots \quad (4.1.10)$$

where $c_m = c_{m+i}$ if $q_i \neq 0$, and β is such that

$$1 = \sum_0^{\infty} \frac{q_n\beta^n}{P_n} = \sum_0^{\infty} \frac{q_n}{c_n}.$$

If the greatest common divisor (g.c.d.) of the indices i for which $q_i \neq 0$, is unity then $c_n = c = 1$ for all n, in which case the probabilities for X are

$$p_0 = 0,\ p_1 = 1-\beta,\ p_2 = (1-\beta)\beta,\ p_3 = (1-\beta)\beta^2, \ldots \qquad \ldots \quad (4.1.11)$$

which is the truncated geometric distribution.

Let the g.c.d. of the indices i for which $q_i \neq 0$ be 2. Then the general solution (4.1.11) gives the probabilities

$$p_i = \begin{cases} 0 & \text{if } i = 0 \\ \beta^{i-1}(1-c\beta) & \text{if } i \text{ is odd} \\ \beta^{i-1}(c-\beta) & \text{if } i \text{ is even} \end{cases} \qquad \ldots \quad (4.1.12)$$

where c and β are such that the expressions in (4.1.12) are nonnegative and less than unity. Other distributions are possible depending on the support of the Y distribution characterized by q_i.

If X and Y are required to have the same distribution i.e., $p_i = q_i$ for all i in (4.1.9), then it is easily seen by substituting the general solution (4.1.10) in (4.1.9) that

$$p_{gi} = q_{gi} = (1-\beta^g)\beta^{g(i-1)}, \quad i = 1, 2, \ldots \qquad \ldots \quad (4.1.13)$$

where g is the g.c.d. of the indices i for which $p_i \neq 0$.

The solutions (4.1.11–4.1.13) for the discrete (lattice) case provide a complete answer to a question raised by Grosswald and Kotz (1980). The solution (4.1.8) for the non-lattice case is obtained by these authors under some regularity conditions on the probability measure μ.

4.2. *Record value problem* : A random variable $X_{(n)}$ is said to be the n-th record value in successive draws of independent observations from a population if $X_{(n)} > X_{(n-1)}$ for $n > 1$ and $X_{(1)} = X_1$, the first observation. If $X \sim E(\lambda)$, then $[X_{(n)} - X_{(n-1)}] \sim E(\lambda)$. We shall find the class of distributions for which this property holds.

Let F denote the p.d.f. of a nonnegative random variable X and define $G = (1-F)$. Further let H_{n-1} denote the p.d.f. of $X_{(n-1)}$. If $X_{(1)}$ and $X_{(n)} - X_{(n-1)}$ have the same distribution, then

$$\int \frac{G(y+x)}{G(y)} \, dH_{n-1}(y) = G(x). \qquad \dots \quad (4.2.1)$$

An application of Theorem 3.2 to (4.2.1) shows that

$$G(x) = p(x)e^{-\lambda x} \qquad \dots \quad (4.2.2)$$

where $p(x) = p(x+z)$ for all z belonging to the support of G_{n-1} which is the same as the support of G. The distribution function (4.2.2) is $E(\lambda)$ if the support of G is not a lattice. The characterization of $E(\lambda)$ through the equation (4.2.1) has been obtained by Ahsanullah (1978, 1979) under some conditions on the hazard rate associated with G.

4.3. *Conditional expectation* : Sahobov and Geshev (1974) established that for a nonnegative random variable X,

$$E[(X-z)^k \mid X \geqslant z] = E(X^k) \; \forall \, z \geqslant 0 \Longrightarrow X \sim E(\lambda). \qquad \dots \quad (4.3.1)$$

We prove this result by an application of Theorem 3.2.

Let F be the d.f. of X and define $G = 1-F$. Then the equation (4.3.1) is equivalent to

$$\int_z^\infty (y-z)^k dF(y) = G(z)E(X^k) = c\, G(z) \qquad \dots \quad (4.3.2)$$

which is equivalent to

$$\int_z^\infty (y-z)^{k-1} G(y) dy = \frac{c}{k} G(z).$$

or

$$\int_0^\infty G(y+z) y^{k-1} dy = \frac{c}{k} G(z). \qquad \dots \quad (4.3.3)$$

An application of Theorem 3.2 shows that

$$G(x) = e^{-\lambda x},$$

i.e., $X \sim E(\lambda)$. If X and z are allowed to take only integral values, then X has the geometric distribution.

4.4. *Order statistics* : Let $X_{i,n}$ denote the i-th order statistic in a sample of n observations. Puri and Rubin (1970) considered the problem of characterizing the distribution of X by the property that $X_{2,2}-X_{1,2}$ has the same distribution as X. Rossberg (1972) and more recently Ramachandran (1980) considered the more general problem of identical distribution of $X_{i+1,n}-X_{i,n}$ and $X_{1,n-i}$. We consider the Puri-Rubin problem as the more general problem can be solved in exactly the same way.

Defining F and G as before, the condition that $X_{2,2}-X_{1,2}$ and X have the same distribution gives the equation

$$2\int_0^\infty G(x+y)dF(y) = G(x), \; x \geqslant 0. \qquad \dots \quad (4.4.1)$$

Applying Theorem 3.2, we find

$$G(x) = p(x)e^{-\lambda x}$$

where $p(x) = p(x+z) \; \forall \; z \in \underline{F}$, provided $F(0) < 1/2$. If the support of F is not a lattice then $X \sim E(\lambda)$. If the support of F is the set of integers, then

$$G(x) = c\beta^x, \; x = 0, 1, \dots$$

which yields the probabilities

$$p_0 \lessdot \frac{1}{2}, p_1 = c(1-\beta), \; p_2 = c\beta(1-\beta), \dots$$

where p_0 and c are chosen such that $p_0+p_1+\dots = 1$. If $p_0 = 1/2$, then one of the equations to be satisfied is

$$p_1p_2+p_2p_3+\dots = 0$$

which implies that only one other $p_i = 1/2$ and the rest are zero.

5. Characterizations of the Pareto distribution

The Pareto law which plays an important role in the study of income distribution specifies the p.d.f. as

$$F(x) = \begin{cases} 1-a^k x^{-k}, & x \geqslant a, \\ 0, & x < a. \end{cases}$$

It is seen that the transformation $x = ae^y$ makes y an exponential variable. Then, for every characterization of the exponential law, we can state the corresponding characterization of the Pareto law. Some of the interesting results obtained by using the Theorems of Section 4 are as follows.

Theorem 5.1 : *Consider a nonnegative random variable X truncated at $a > 0$, and let R be an independent random variable distributed over the interval* (0, 1). *If the distribution of $Y = XR$ truncated at a is the same as that of X, then X has the Pareto distribution.*

Note. If X represents the income of an individual and R the proportion of the income reported, then the distributions of the actual income and the reported income both truncated below at a given point will be the same only if the actual income has the Pareto distribution. Krishnaji (1970) established this result when R has a uniform distribution. Our theorem shows the validity of this result for any nontrivial arbitrary distribution of R on (0, 1).

Theorem 5.2 : *Let X be a nonnegative random variable and $h(\cdot)$ be a nonnegative and nondecreasing function such that $h(0) = 0$ and for any $a > 0$*

$$E\left[h\left(\frac{X}{a}\right) | X \geqslant a\right] = \textit{constant};$$

then X has the Pareto distribution.

While the paper was in Prese the authors have come to know that Theorem 3.1 of this paper had been proved by J. Deny in 1960 under a more general set up.

References

Aczél, J. (1966) : *Lectures on Functional Equations and Their Applications.* Academic Press, New York.

Ahsanullah, M. (1978) : Record values and the exponential distribution. *Ann. Inst. Stat. Math.*, **30**, 429-433.

——— (1979) : Characterization of the exponential distribution by record values. *Sankhyā* *B* **41**, 116-121.

Brandhofe, T. and Davies, L. (1980) : On a functional equation in the theory of linear statistics. *Ann. Inst. Statist. Math.*, **32**, 17-23.

CHOQUET, G. and DENY, J. (1960) : Sur l'equation de convolution $\mu = \mu * \sigma$. *Compt. Rend.* 250, 799-801.

FELLER, W. (1971) : *An Introduction to Probability Theory and its Applications.* Vol. II (Second Edition), Wiley, New York.

GALAMBOS, J. and KOTZ, S. (1978) : *Characterizations of Probability Distributions.* Lecture Notes in Mathematics, No. 675. Springer-Verlag.

GROSSWALD, E. and KOTZ, S. (1980) : An integrated lack of memory characterization of the exponential distribution. Tech. Rept. Temple University, Philadelphia.

HUANG, J. S. (1978) : On a "lack of memory" property. *Statistical Series* 1978-84, University of Guelph, Canada.

KRISHNAJI, N. (1970) : Characterization of the Pareto distribution through a model of unreported incomes. *Econometrica,* **38**, 251-255.

PURI, P. S. and RUBIN, H. (1970) : A characterization based on the absolute difference of two i.i.d. random variables. *Ann. Math. Statist.*, **41**, 2113-2122.

RAMACHANDRAN, B. and RAO, C. R. (1970) : Solutions of functional equations arising in some regression problems and a characterization of the Cauchy law. *Sankhyā A* **32**, 1-30.

RAMACHANDRAN, B. (1979) : On the "strong memorylessness property" of the exponential and geometric probability laws. *Sankhyā A* **41**, 244-251.

——— (1980) : An integral equation in probability theory and its implications. Tech. Rept. Indian Statistical Institute, New Delhi.

RAO, C. R. (1965) : On discrete distributions arising out of methods of ascertainment. *Sankhyā A* **27**, 311-324.

RAO, C. R. and RUBIN, H. (1964) : On a characterization of the Poisson distribution. *Sankhyā A* **26**, 294-298.

ROSSBERG, H. J. (1972) : Characterization of the exponential and Pareto distributions by means of some properties of the distributions which the differences and quotients of order statistics are subject to. *Math. Operationsforsch. u. Statist.*, **3**, 207-216.

SHANBHAG, D. N. (1977) : An extension of the Rao-Rubin characterization. *J. Appl. Prob.*, **14**, 640-646.

SHIMIZU, R. (1978) : Solution to a functional equation and its application to some characterization problems. *Sankhyā A* **40**, 319-332.

SOHOBOV, O. M. and GESHEV, A. A. (1974) : Characteristic property of the exponential distrition. *Natura. Univ. Plovdiv.*, **7**, 25-28 (in Russian).

Paper received : April, 1981.
Revised : January, 1982.